Some Worked Examples in Part II Tripos

GR

Preliminary. Some of the following examples concern the Schwarzschild Lagrangian in the equatorial plane $\theta = \frac{\pi}{2}$ is

$$L = -\left(1 - \frac{r_s}{r}\right) c^2 \dot{t}^2 + \left(1 - \frac{r_s}{r}\right)^{-1} \dot{r}^2 + r^2 \dot{\phi}^2 \tag{1}$$

with first integrals

$$\dot{t}\left(1 - \frac{r_s}{r}\right) = \frac{E}{c^2} \tag{2}$$

$$r^2 \dot{\phi} = h. \tag{3}$$

Substituting (2) and (3) in $L = k$ where constant $k = -c^2$, 0, or 1 depending on the nature of the geodesic, we obtain

$$\frac{1}{2}\dot{r}^2 + V(r) = \frac{E}{2c^2} \tag{4}$$

where $2V(r) = \left(1 - \frac{r_s}{r}\right)\left(-k + \frac{h^2}{r^2}\right)$.

Circular orbits (S2Q2)

Differentiating (4) gives $\ddot{r} + V'(r) = 0$, so a circular orbit possibly exists at $r = r_0$ where $V(r_0) = \frac{E}{2c^2}$ and

$$V'(r_0) = 0. \tag{5}$$

By substituting perturbation $r = r_0 + \epsilon$ into (4) and using $V(r) \approx V(r_0) + \frac{\epsilon^2}{2} V''(r_0)$, we obtain after differentiating

$$\ddot{\epsilon} + V''(r_0)\epsilon = 0$$

so for stability we must have $V''(r_0) > 0$.

Solving (5) gives

$$\frac{h}{c} = \frac{r_0}{\sqrt{\frac{2r_0}{r_s} - 3}}$$

so a circular orbit is possible only for $r_0 > \frac{3}{2} r_s$.

Requiring stability or instability $V''(r_0) \gtrless 0$ means $\frac{3r_s - r_0}{2r_0 - 3r_s} \lessgtr 0$.

Perihelion precession (§2.8)

With $k = -c^2$, $u = \frac{1}{r}$ and $\frac{d}{d\tau} = \dot{\phi}\frac{d}{d\phi} = hu^2\frac{d}{d\phi}$ from (3), upon differentiating with respect to ϕ, equation (4) gives

$$u'' + u = \frac{1}{l} + \frac{3}{2}r_s u^2 \tag{6}$$

where $l = \frac{2h^2}{r_s c^2}$, and $\frac{r_s}{l} \ll 1$ is assumed.

The first order solution is the Kepler orbit ($r_s = 0$)

$$u_0 = \frac{1 + e\cos\phi}{l}$$

where $0 \leqslant e < 1$ for bound orbits.

Substituting higher order solutions $u = u_0 + \frac{r_s}{l}u_1 + \mathcal{O}\left(\frac{r_s}{l}\right)^2$ into (6) gives

$$\begin{aligned} u_1'' + u_1 &= \frac{3l}{2}u_0^2 + \mathcal{O}\left(\frac{r_s}{l}\right) \\ &\approx \frac{3}{2l}\left(\underbrace{\cancel{1 + \frac{1}{2}e^2 + \frac{1}{2}e^2\cos 2\phi}}_{\text{small } 2\pi-\text{periodic motion}} + \underbrace{2e\cos\phi}_{\text{resonance}}\right). \end{aligned}$$

Considering the non-periodic solution to the resonance term, we obtain $u_1 = \frac{3e}{2l}\phi\sin\phi$, hence

$$\begin{aligned} u &\simeq \frac{1}{l} + \frac{e}{l}\left(\cos\phi + \frac{3r_s}{2l}\phi\sin\phi\right) \\ &\simeq \frac{1}{l} + \frac{e}{l}\cos\left(\phi - \frac{3r_s}{2l}\phi\right) \end{aligned}$$

so perihelion advances by $2\pi\frac{3r_s}{2l}$ radians per orbit.

Deflection of light (§2.9)

With $k = 0$, $u = \frac{1}{r}$ and $\frac{d}{d\tau} = \dot{\phi}\frac{d}{d\phi} = hu^2\frac{d}{d\phi}$ from (3), upon differentiating with respect to ϕ, equation (4) gives

$$u'' + u = \frac{3}{2}r_s u^2 \tag{7}$$

with first order solution ($r_s = 0$) $u = \frac{1}{b}\sin\phi$ where b is the distance of closest approach in the absence of the gravitational field.

Substituting higher order solutions $u = u_0 + \frac{r_s}{b}u_1 + \mathcal{O}\left(\frac{r_s}{b}\right)^2$ into (7) gives

$$u_1'' + u_1 = \frac{3}{4b}(1 - \cos 2\phi)$$

whence

$$u = \frac{1}{b}\sin\phi + \underbrace{\frac{r_s}{4b^2}(3+\cos 2\phi)}_{\text{particular solution}} + \underbrace{\frac{r_s}{b^2}\cos\phi}_{\text{complementary solution}}$$

so $u \to 0$ as $\phi \to \pi$. As $\phi \to -\Delta\phi$ where $\Delta\phi$ is the small angle of deflection, $u \to 0$ so

$$0 \simeq -\frac{\Delta\phi}{b} + \frac{2r_s}{b^2}$$

giving $\Delta\phi \approx \frac{2r_s}{b}$.

Deflection of massive objects (S2Q4)

Equation (6) applies substitution of $u = u_0 + \epsilon u_1 + \mathcal{O}(\epsilon^2) \approx \frac{1}{b}\sin\phi + \epsilon u_1$ yields

$$\epsilon u_1'' + \epsilon u_1 = \underbrace{\frac{r_s c^2}{2h^2}}_{\sim b\epsilon} + \frac{3}{2}r_s(\frac{1}{b}\sin\phi + \epsilon u_1)^2$$

so

$$u_1'' + u_1 = \frac{bc^2}{2h^2} + \frac{3}{2b}\sin^2\phi$$

giving a solution

$$u = \frac{1}{b}\sin\phi + \epsilon\left[\frac{bc^2}{2h^2} + \frac{1}{4b}(3+\cos 2\phi) + \underbrace{\left(\frac{1}{b} + \frac{bc^2}{2h^2}\right)\cos\phi}_{\text{complementary solution}}\right]$$

such that $u \to 0$ as $\phi \to \pi$. As $\phi \to -\Delta\phi$ where $\Delta\phi$ is the small angle of deflection, $u \to 0$ so

$$0 \simeq -\frac{\Delta\phi}{b} + 2\epsilon\left(\frac{1}{b} + \frac{bc^2}{2h^2}\right)$$

giving $\Delta\phi \approx 2\epsilon\left(1 + \frac{b^2c^2}{2h^2}\right)$.

Killing vector fields (S3Q4 and S3Q8)

Consider an isometry $x \mapsto y$ under which ds^2 is invariant:

$$g_{ab}(y)\frac{\partial y^a}{\partial x^c}\frac{\partial y^b}{\partial x^d} = g_{cd}(x) \tag{8}$$

so writing $y^a = x^a + \epsilon\xi^a(x)$, whence $\frac{\partial y^a}{\partial x^c} = \delta^a_c + \epsilon\xi^a{}_{,c}$ and $g_{ab}(y) = g_{ab}(x) + \epsilon g_{ab,c}\xi^c(x) + o(\epsilon)$, we have by comparing $\mathcal{O}(\epsilon)$ terms in (8)

$$\xi^c g_{ab,c} + g_{cb}\xi^c{}_{,a} + g_{ca}\xi^c{}_{,b} = 0$$

which in local inertial coordinates, or upon direct computation, reduces to the Killing equation

$$\xi_{b;a} + \xi_{a;b} = 0.$$

Now first consider $R^a{}_{b(cd)} = 0 \Rightarrow R^a{}_{(bcd)} = 0$ but by Bianchi's first identity $R^a{}_{[bcd]} = 0$ so

$$R^d{}_{acb} + R^d{}_{bac} + R^d{}_{cba} = 0. \tag{9}$$

By the Ricci identity $\xi_{c;ba} - \xi_{c;ab} = -R^d{}_{cab}\xi_d$ so

$$\begin{aligned}
\xi_{[a;bc]} &= \frac{1}{6}\left(\xi_{a;bc} - \xi_{a;cb} + \xi_{b;ca} - \xi_{b;ac} + \xi_{c;ab} - \xi_{c;ba}\right) \\
&= \frac{1}{6}\left(-R^d{}_{acb} - R^d{}_{bac} - R^d{}_{cba}\right)\xi_d \\
&= 0
\end{aligned}$$

by (9).

Mixed Symmetry of the Riemann Tensor (S3Q6 or §3.9)

In local inertial coordinates

$$0 = \left(g^{ab}g_{bc}\right)_{,d} = g_{bc}g^{ab}{}_{,d} \Rightarrow g^{ab}{}_{,c} = 0$$

and thus

$$\Gamma^a{}_{bc,d} = \frac{1}{2}g^{ae}\left(g_{be,cd} + g_{ce,bd} - g_{bc,ed}\right).$$

Hence

$$\begin{aligned}
R_{abcd} &= g_{ae}R^e{}_{bcd} \\
&= g_{ae}\left(\Gamma^e{}_{bd,c} - \Gamma^e{}_{bc,d}\right) \\
&= \frac{1}{2}\left(g_{da,bc} - g_{bd,ac} + g_{ca,bd} - g_{bc,ad}\right) \\
&= R_{cdab}
\end{aligned}$$

by symmetry of partial derivatives and the metric tensor.

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Rayleigh waves (§3.2.6)

This is a combination of self-sustained P and SV waves at the boundary.

Ansätz are

$$\begin{aligned}
\varphi &= Ae^{i(kx-\omega t)+\alpha y}, \quad \alpha = \sqrt{k^2 - \frac{\omega^2}{c_p^2}},\ \mathbf{k}_p = k\hat{\mathbf{x}} - i\alpha\hat{\mathbf{y}} \\
\underline{\psi} &= B\hat{\mathbf{z}}e^{i(kx-\omega t)+\beta y}, \quad \beta = \sqrt{k^2 - \frac{\omega^2}{c_s^2}},\ \mathbf{k}_s = k\hat{\mathbf{x}} - i\beta\hat{\mathbf{y}} \\
\Rightarrow \mathbf{u} &= \left(ikAe^{\alpha y} + \beta Be^{\beta y},\ \alpha Ae^{\alpha y} - ikBe^{\beta y}\right) e^{i(kx-\omega t)}
\end{aligned}$$

subject to boundary conditions $\sigma_{xy} = \sigma_{yy} = 0$ so eliminating A, B yields the dispersion relation.

Love waves (§4.1.2)

This is an SH wave trapped in a layer between $z = 0$ (medium interface) and $z = h$ (free surface), so seek a solution

$$\mathbf{u} = (0,\ 1,\ 0)f(z)e^{i(kx-\omega t)}$$

which must satisfy the wave equation so

$$f(z) = \begin{cases} A\cos\left[m_1(h-z)\right] & 0 < z < h \\ Be^{m_2 z} & z < 0 \end{cases}$$

where respectively

$$\begin{cases} m_1 = k\left(\frac{c^2}{c_1^2} - 1\right)^{\frac{1}{2}} & 0 < z < h \\ m_2 = k\left(1 - \frac{c^2}{c_2^2}\right)^{\frac{1}{2}} & z < 0 \end{cases}$$

where $c_1 < c < c_2$.

Now continuity of displacement and stress requires that $u|_-^+ = \mu\frac{\partial u}{\partial z}\Big|_-^+ = 0$ at $z = 0$, which gives the dispersion relation.

Waves approaching a beach (§5.1.7)

Omitted, but you should be familiar with this example. The key is that some components of the wave-vector are constant, and so is the local frequency, from which $k(x)$ is determined by $h(x)$.

Mach cones (§5.3.3)

With $\mathbf{U} = (Mc_0,\ 0,\ 0)$, the dispersion relation becomes

$$\omega_s = c_0 k - \mathbf{U}\cdot\mathbf{k} = c_0 k(1 - M\cos\phi)$$

where ϕ is the angle between $\mathbf{U}$ and $\mathbf{k}$.

Steady waves are possible when $\phi = \arccos \frac{1}{M}$, which implies

$$\hat{\mathbf{k}} = (\cos\phi,\, \sin\phi,\, 0) = \left(\frac{1}{M},\, \left(1 - \frac{1}{M^2}\right)^{\frac{1}{2}}, 0\right)$$

and

$$\mathbf{c}_{gs} = c_0\hat{\mathbf{k}} - \mathbf{U} = c_0\sqrt{M^2 - 1}(-\sin\phi,\, \cos\phi).$$

Noting that $\mathbf{c}_{gs} \cdot \mathbf{U} < 0$ and $\mathbf{c}_{gs} \cdot \hat{\mathbf{k}} = 0$, we deduce there are steady waves in a cone with semi-angle

$$\alpha = \arcsin\cos\phi = \arcsin\frac{1}{M}.$$

Ship/duck waves (§5.3.4)

Taking $\mathbf{U} = (U,\, 0,\, 0)$ and $\mathbf{k} = k(\cos\phi,\, 0,\, \sin\phi)$, the dispersion relation becomes

$$\left(\omega_{(s)} + \mathbf{U}\cdot\mathbf{k}\right)^2 = gk$$

so solving for

$$\omega = \Omega(k_1,\, k_3) = \pm\sqrt{gk} - Uk\cos\phi$$

gives

$$\mathbf{c}_{gs} = \pm\frac{1}{2}\sqrt{\frac{g}{k}}\mathbf{k} \hat{-} \mathbf{U}.$$

Steady waves require

$$U\cos\phi = \pm\sqrt{\frac{g}{k}}$$

so $\mathbf{c}_g = \frac{U}{2}(\cos^2\phi - 2,\, 0,\, \sin\phi\cos\phi)$ and $\mathbf{c}_g \cdot \hat{\mathbf{x}} < 0$. Besides, $\Omega_t = 0$ and $\nabla_{\mathbf{x}}\Omega = 0$ so rays are straight.

On such rays let

$$\frac{dz}{dx} = -\tan\psi$$

then by ray-tracing equations

$$\tan\psi = -\frac{\frac{\partial\omega}{\partial k_3}}{\frac{\partial\omega}{\partial k_1}} = -\frac{(\mathbf{c_g})_z}{(\mathbf{c_g})_x} = \frac{\tan\phi}{1 + 2\tan^2\phi}.$$

So solving $\frac{\partial\tan\psi}{\partial\tan\phi} = 0$ yields the maximum angle

$$\tan|\psi| = \frac{1}{2\sqrt{2}}.$$

Now $\theta = \mathbf{k}\cdot\mathbf{x} = kr\cos(\pi-\psi-\phi) = -kr\frac{\cos\psi}{\cos\phi}\left(\frac{1}{1+2\tan^2\phi}\right)$, hence

$$\begin{aligned} x &= -r\cos\psi \\ z &= -x\tan\psi \end{aligned}$$

gives an parametric description in terms of $(\theta,\ \phi)$.

FD II

Flow past a sphere (§4.2 and S2Q6) and the rise of a spherical bubble (§7.8)

By linearity and spherical symmetry

$$\begin{aligned} \mathbf{u} &= \mathbf{U}f(r) + \mathbf{x}\left(\mathbf{U}\cdot\mathbf{x}\right)g(r) \\ p &= \mu(\mathbf{U}\cdot\mathbf{x})h(r) \end{aligned}$$

so substitution into incompressibility condition and the Stokes equation yields, after much algebra,

$$\frac{f'}{r} + 4g + rg' = 0, \quad f'' + 2\frac{f'}{r} + 2g = h \text{ and } g'' + \frac{6g'}{r} = \frac{h'}{r}.$$

Eliminating $f,\ h$ gives an equidimensional equation for g,

$$r^2g''' + 11rg'' + 24g' = 0$$

with solutions $f = -\frac{\alpha+4}{\alpha+2}r^{\alpha+2}$, $g = r^\alpha$ and $h = -(\alpha+5)(\alpha+2)r^\alpha$ where $\alpha = 0,\ -3,\ -5$. Hence the general solution is

$$\begin{aligned} \mathbf{u} &= \mathbf{U}\left(-2Ar^2 + B + Cr^{-1} - \frac{1}{3}Dr^{-3}\right) + \mathbf{x}\left(\mathbf{U}\cdot\mathbf{x}\right)\left(A + Cr^{-3} + Dr^{-5}\right) \\ p &= \mu(\mathbf{U}\cdot\mathbf{x})\left(-10A + 2Cr^{-3}\right) \\ \sigma\cdot\mathbf{u} &= U\left(-3Ar + 2Dr^{-4}\right) + \mathbf{x}\left(\mathbf{U}\cdot\mathbf{x}\right)\left(9Ar^{-1} - 6Cr^{-4} - 6Dr^{-6}\right) \end{aligned}$$

and with imposed boundary condition on a sphere, we have

$$A = 0, \quad B = 1, \quad C = -\frac{3}{4}a \quad \text{and} \quad D = \frac{3}{4}a^3$$

with drag force on the sphere

$$\int_{r=a} \sigma\cdot\mathbf{n}dS = 6\pi\mu a\mathbf{U};$$

and on a bubble,

$$A = 0, \quad B = 1, \quad ,C = -\frac{a}{2} \text{ and } D = 0$$

with drag force on the bubble

$$\int_{r=a} \sigma\cdot\mathbf{n}dS = 4\pi\mu a\mathbf{U}.$$

For the latter, $\mathbf{u} = -\frac{a^3}{2r^3}\mathbf{U} + \frac{3a^3}{2r^5}\mathbf{x}\,(\mathbf{U}\cdot\mathbf{x})$ and $\mathbf{u}\cdot\mathbf{n} = \mathbf{U}\cdot\mathbf{n}$, with dissipation calculated (after much algebra) to be $12\pi\mu a\mathbf{U}^2$.

Flow in a corner (§4.4 and S2Q8)

For a Stokes flow, $\underline{\omega}$ is harmonic and $\underline{\omega} = (0,\ 0,\ -\nabla^2\psi)$ so

$$\nabla^2\nabla^2\psi = \left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\phi^2\right)^2 \psi = 0$$

solved by a power law relation $\psi = r^\lambda f(\phi)$:

$$r^{\lambda-4}\left(\frac{d^2}{d\phi^2} + \lambda^2\right)\left[\frac{d^2}{d\phi^2} + (\lambda-2)^2\right] f = 0$$

where boundary conditions must be satisfied, thus determining λ and f.

To calculate the pressure, it is most convenient to employ the *Stokes equation*, written in the form

$$\nabla\frac{p}{\mu} = \nabla^2\mathbf{u} = \nabla(\nabla\cdot\mathbf{u}) - \nabla\times\nabla\times\mathbf{u} = -\nabla\times\underline{\omega}.$$

You may find there is a log-type singularity in pressure, because of a perfectly sharp corner at $r = 0$. But in reality this is impossible; on the other hand, such singularity is integrable so all forces remain finite.

In the far field pressure and velocity become unbounded, because the Stokes approximation no longer holds, and the inertial terms are not negligible.

Gravitational spreading of droplets (§5.6 and S3Q4)

General remarks. A common trick used in thin layer flows/lubrication theory is to integrate the incompressibility condition with appropriate boundary conditions to show the volume flux, or some other quantity, is conserved.

2-d case (S3Q4).

Have governing equations

$$\begin{aligned}\frac{\partial p}{\partial x} &= \mu\frac{\partial^2 u}{\partial y^2}\\ 0 &= -\frac{\partial p}{\partial y} - \rho g\end{aligned}$$

subject to boundary conditions

$$u|_{y=0} = 0, \quad \left.\frac{\partial u}{\partial y}\right|_{y=h} = 0 \text{ and } p|_{y=h} = p_0$$

giving $p = p_0 + \rho g(h - y)$ and thus $\frac{\partial p}{\partial x} = \rho g \frac{\partial h}{\partial x}$, and thus

$$\frac{\partial^2 u}{\partial y^2} = \frac{\rho g}{\mu}\frac{\partial h}{\partial x}$$

from which u and Q can be calculated in terms of $\frac{\partial h}{\partial x}$.

By conservation of mass,

$$-\frac{\partial h}{\partial t}dx = Q(x + dx) - Q(x)$$

which yields a partial differential equation for $h(x,\, t)$.

By conservation of volume $V = \int_{-a}^{a} h(x,\, t)dx$ and scale relation $V = Ha$, we could now seek a similarity solution

$$h(x,\, t) = H(t)f(\frac{x}{a}).$$

3-d case (§5.6).

Have governing equations

$$\begin{aligned} \frac{\partial p}{\partial r} &= \mu\frac{\partial^2 u}{\partial z^2} \\ 0 &= -\frac{\partial p}{\partial z} - \rho g \end{aligned}$$

subject to boundary conditions

$$u|_{z=0} = 0, \quad \left.\frac{\partial u}{\partial z}\right|_{z=h} = 0 \text{ and } \; p|_{z=h} = p_0$$

giving $p = p_0 + \rho g(h - z)$ and thus $\frac{\partial p}{\partial r} = \rho g \frac{\partial h}{\partial r}$, and thus

$$\frac{\partial^2 u}{\partial z^2} = \frac{\rho g}{\mu}\frac{\partial h}{\partial r}$$

from which u and Q can be calculated in terms of $\frac{\partial h}{\partial r}$.

By conservation of mass,

$$\frac{\partial}{\partial r}2\pi r Q(r) + 2\pi r\frac{\partial h}{\partial t} = 0$$

which yields a partial differential equation for $h(r,\, t)$.

By conservation of volume $V = \int_0^a 2\pi r h(r,\, t)dr$ and scale relation $V = Ha^2$, we could now seek a similarity solution

$$h(x,\, t) = H(t)f(\frac{r}{a(t)}).$$

2-d momentum jets (§7.4)

The key is to find a conserved quantity, in this case the momentum flux, so as to give a constraint on the layer thickness $\delta(x)$ and the typical speed $U(x)$, and provide a boundary condi-

tion.

The trick is re-use the boundary layer equation when differentiating the momentum flux in order to show it is conserved. For more details please see the lecture notes.